§1? ?: Triple Integrals

IDEA: Integrate functions of three variables.
Remark: All the hard work to "up" the dimension is already done.
* 1 variable → 2 variables was the hardest part *

Conceptually, this is no different from double integrals (pictures are harder).

$$\iiint_R f(x,y,z)\,dV$$

is computable via an iterated integral...
↳ same principle as before, the order of integration is more-or-less up to us, as long as we parameterize appropriately.

ex) compute $\iiint_E (xy+z^3)\,dV$ for $E=[0,2]\times[0,1]\times[0,3]$

Sol: $= \int_{x=0}^{2}\int_{y=0}^{1}\int_{z=0}^{3} (xy+z^3)\,dz\,dy\,dx$ $\quad 0\le x\le 2$, $0\le y\le 1$, $0\le z\le 3$

innermost (z): $\int_{z=0}^{3} xy+z^3\,dz$

$$= \left[xyz + \frac{1}{3}z^3 \right]_{z=0}^{3}$$

$$= (3xy + 9) - 0$$

<u>middle integral (y):</u>

$$\int_{y=0}^{1} 3xy + 9 \, dy$$

$$= \left[\frac{3}{2}xy^2 + 9y \right]_{y=0}^{1}$$

$$= \left(\frac{3}{2}x + 9 \right) - 0$$

<u>outer integral (x):</u>

$$\int_{x=0}^{2} \frac{3}{2}x + 9 \, dx$$

$$= \left[\frac{3}{4}x^2 + 9x \right]_{x=0}^{2}$$

$$= \left(\frac{3}{4}(4) + 18 \right) - 0$$

$$= 21$$

$$\iiint_E (xy + z^2)\, dV = 21$$

ex) Compute $\iiint_R (2x-y)\, dV$ where
$R = \{(x,y,z) : 0 \le z \le 2,\ 0 \le y \le z^2,\ 0 \le x \le y-z\}$

notice: this parametrization has the form:
$$\{(x,y,z) : c_1 \le z \le c_2,\ g_1(z) \le y \le g_2(z),\ h_1(y,z) \le x \le h_2(y,z)\}$$
this has the same form as when we computed double integrals ("what we liked").

$$\left\{(x,y,z) : \begin{array}{l} c_1 \le z \le c_2, \\ g_1(z) \le y \le g_2(z), \\ h_1(y,z) \le x \le h_2(y,z) \end{array}\right\}$$
# of variables increases

Sol: $\iiint_R (2x-y)\, dV$
$$= \int_{z=0}^{2} \int_{y=0}^{z^2} \int_{x=0}^{y-z} (2x-y)\, dx\, dy\, dz =$$

inner (x):
$$\int_{x=0}^{y-z} 2x-y\, dx$$
$$= \left[x^2 - xy\right]_{x=0}^{y-z}$$
$$= \left((y-z)^2 - (y-z)y\right) - 0$$
$$= \cancel{y^2} - 2yz + z^2 - \cancel{y^2} + yz$$
$$= z^2 - yz$$

Middle (y):

$$\int_{y=0}^{z^2} (z^2 - yz)\, dy$$

$$= \left[yz^2 - \tfrac{1}{2}y^2 z \right]_{y=0}^{z^2}$$

$$= \left(z^2 \cdot z^2 - \tfrac{1}{2}(z^2)^2\right) - 0$$

$$= z^4 - \tfrac{1}{2} z^5$$

Outermost (z):

$$\int_{z=0}^{2} z^4 - \tfrac{1}{2} z^5\, dz$$

$$= \left[\tfrac{1}{5} z^5 - \tfrac{1}{12} z^6 \right]_{z=0}^{2}$$

$$= \tfrac{1}{5}(32) - (64)\left(\tfrac{1}{12}\right) - 0$$

$$= \frac{16}{15}$$

$$\boxed{\iiint_R (2x - y)\, dV = \frac{16}{15}}$$

Remark on reparametrization:

To change the order of integration, we must reparametrize to look like the form from earlier (innermost has multiple variables).

for this region R in the previous example, to change the order to $dy\,dx\,dz$:

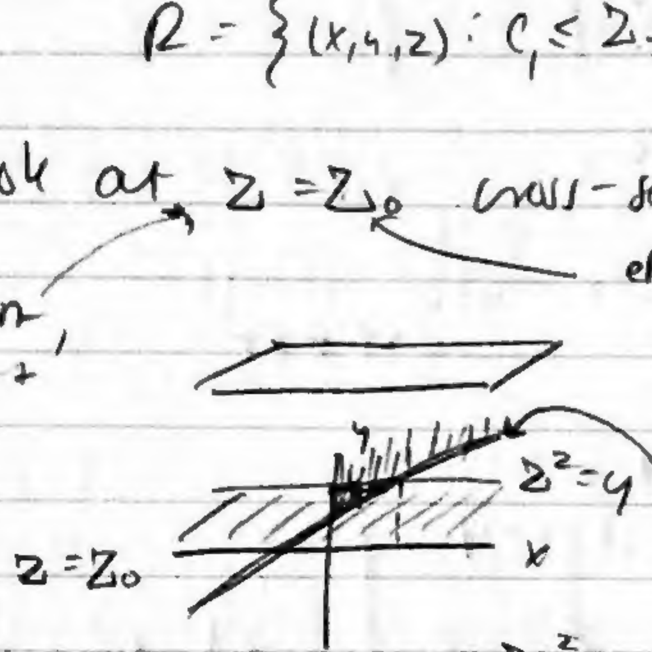
reparametrize the form:

$$R = \{(x,y,z) : c_1 \le z \le c_2,\ g_1(z) \le x \le g_2(z),\ h_1(x,z) \le y \le h_2(x,z)\}$$

look at $z = z_0$ cross-section

common constant

effectively fixing z as constant



$x = y - z$

$y = x + z$

$z^2 = y = x + z$

$\therefore x = z^2 - z$

based on picture $\begin{cases} 0 \le x < z^2 - z \\ x + z \le y \le z^2 \end{cases}$

$$\therefore R = \left\{(x,y,z) : \begin{array}{l} 0 \le z \le 2, \\ 0 \le x \le z^2 - z, \\ x + z \le y \le z^2 \end{array}\right\}$$

↳ this is equivalent to the original region, but reparametrized. ($dy\,dx\,dz$ from $dx\,dy\,dz$)

ex) Compute the volume of the tetrahedron T with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$.

Sol: $\mathrm{Vol}(T) = \iiint_T 1 \, dV$

xy-shadow

$m = \frac{1-0}{0-1} = -1$

$y - 0 = -1(x-1)$

$y = 1-x$

$0 \le x \le 1$

$0 \le y \le 1-x$

$0 \le z \le 1-x-y$

To the plane:

$\vec{u} \times \vec{v} = \vec{n}$

$p = (0,0,1)$

$\langle 1,0,-1 \rangle = \vec{u}$

$\langle 0,1,-1 \rangle = \vec{v}$

$\vec{n} = \langle 1,0,-1 \rangle \times \langle 0,1,-1 \rangle$

$= \begin{vmatrix} i & j & k \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \langle 1,1,1 \rangle$

$\therefore 0 = \vec{n} \cdot (\vec{x} - \vec{p})$

$0 = \langle 1,1,1 \rangle \cdot \langle x, y, z-1 \rangle$

$0 = x + y + z - 1$

$z = 1 - x - y$

$$\mathrm{Vol}(T) = \int_{x=0}^{1} \int_{y=0}^{1-x} \int_{z=0}^{1-x-y} 1 \, dz\, dy\, dx$$

$$= \int_{x=0}^{1} \int_{y=0}^{1-x} \Big[z \Big]_0^{1-x-y} dy\, dx$$

$$= \int_{x=0}^{1} \int_{y=0}^{1-x} (1-x-y)\, dy\, dx$$

$$= \int_{x=0}^{1} \Big[y - xy - \tfrac{1}{2}y^2 \Big]_{y=0}^{1-x} dx$$

$$= \int_{x=0}^{1} \Big((1-x) - x(1-x) - \tfrac{1}{2}(1-x)^2 \Big) dx$$

$$= \frac{1}{2} \int_{x=0}^{1} (1-x)^2 \, dx$$

$$= \frac{1}{2} \cdot -\frac{1}{3} \Big[(1-x)^3 \Big]_{x=0}^{1} = \frac{1}{6}$$